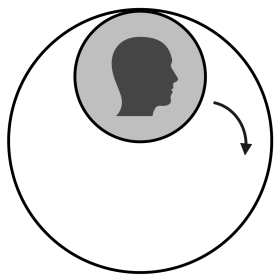
Grade 6 Math Circles
March 5 & 6 & 7, 2024
Math Paradoxes - Problem Set

1. (**Coin Rotation Paradox**) Suppose that there is a circular loop with a coin sitting inside it, as shown in the diagram. The inner radius of the loop is two times the radius of the coin. Let the coin roll without slipping once around the loop. How many turns does the coin make relative to an external stationary observer?

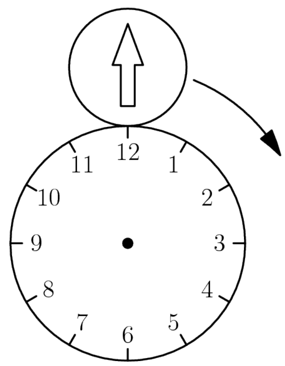


Solution 1: We can find the circumference (length) of the rolling surface (the circular loop that encloses the coin) as a multiple of the coin's circumference. Since the loop has an inner radius that is two times the radius of the coin, then the circumference of the rolling surface is two times that of the coin. As the coin rolls without slipping, it rotates **counter-clockwise** relative to an external reference frame. However, the path is not straight and causes the coin to make one full turn **clockwise**. Therefore, the total number of turns that the coin makes relative to an external reference frame is $2 - 1 = 1$.

Solution 2: We can find the length of the path that the center of the coin travels. This path is a circle with radius r, where r is the radius of the coin. Since the path that the center of the coin travels on has the same radius as the coin itself, then this path has the same circumference as the coin as well. Therefore, the coin makes one full turn.

2. (**Coin Rotation Paradox**) *This problem is taken from 2015 AMC 10A Problem 14.* The diagram below shows the circular face of a clock with radius 20 cm and a circular disk with radius 10 cm externally tangent to the clock at 12 o'clock. The disk has an arrow painted on it, initially point in the upward vertical direction. Let the disk roll clockwise around the clock

face. At what point on the clock face will the disk be tangent (i.e., touching) when the arrow is next pointing in the upward vertical direction?



Solution: To figure this out, let's first determine the number of complete turns that the disk makes as it goes around the clock once. Since the radius of the clock is 20 cm, and this is two times the radius of the disk, then the circumference of the clock must be two times the circumference of the disk. Therefore, the disk makes $2 + 1 = 3$ complete turns relative to a stationary observer as it rolls around the clock (since the rolling path is a circle which adds an additional turn).

Since the disc makes three full rotations as it rolls around the edge of the clock, then it must have made one full rotation when it is tangent to the 4:00 mark on the clock face. This is when the disk is next pointing in the upward vertical direction.

3. (**Will Rogers Phenomenon**) At a university, the numerical grades (out of 100) of two groups of students, R and S, are shown below.

$$R \text{ grades} = \{43, 56, 77, 78, 81, 88, 90, 95\}$$
$$S \text{ grades} = \{65, 70, 78, 82, 83, 85, 86, 88, 90, 94, 99, 100\}$$

(a) Calculate the mean of the grades of the students in R and the mean of the grades of the students in S. You may use a calculator, if you wish.

Solution: We have that

$$\begin{aligned} R \text{ mean grade} &= \frac{43+56+77+78+81+88+90+95}{8} = \frac{608}{8} = 76 \\ S \text{ mean grade} &= \frac{65+70+78+82+83+85+86+88+90+94+99+100}{12} \\ &= \frac{1020}{12} \\ &= 85 \end{aligned}$$

(b) Suppose that one student in group S is to move to group R. In the list above of S grades, circle all the grades that could be moved into R grades such that the mean of both lists increase.

Solution: These numbers should be circled: $78, 82, 83$. In general, any number that lies between the means of the two groups (excluding the means themselves) can be moved into the group with the lower mean as doing so will increase the mean of both groups.

4. (**Simpson's Paradox**) *This problem is adapted from an online blog post based on a news article*[1]. In a fictional country, all adults belong to one of the following groups based on their educational level:

- high school dropouts
- high school graduates with no post-secondary education
- people with some post-secondary education
- people with Bachelor's or higher degrees

Suppose that the median wage for the country has risen by 1%. However, the median wage within each group has all decreased.

(a) Explain why this is an example of Simpson's Paradox.

[1] **Source:** https://blog.revolutionanalytics.com/2013/07/a-great-example-of-simpsons-paradox.html

Solution: This is an example of Simpson's paradox because the trend in median wage is reversed when the population is put into several groups. Within each of the four groupings of people by educational attainment, the median wage has decreased; however, when these groups are combined, an increase in the median wage is seen.

(b) The table below shows the percentage of the population belonging to each group before and after the change in median wage was measured. Based on this data, explain how the overall median wage of the population is seen to increase despite the drop in the median wage within all four groups.

Educational Attainment	Proportion of Country Before	Proportion of Country After
high school dropouts	9%	9%
high school graduates with no post-secondary education	40%	40%
people with some post-secondary education	40%	20%
people with Bachelor's or higher degrees	12%	32%

Solution: People with Bachelor's or higher degrees earn higher wages than others. After the change in median wage was measured, a greater number of people have Bachelor's or higher degrees, leading to higher wages. This effect overshadows the drop in median wage within each category.

5. (**Birthday Problem**) If there are $n = 3$ people in a room, how many pairs of people would you have to check to be sure that no one shares a birthday? How many pairs of people would you have to check if there are $n = 4$ people in the room? What about $n = 23$? Justify your answer.

Solution: If there are 3 people in a room, then you can count the pairs of people by multiplying the number of people (3) by the number of other people that each person can be paired up with $(3 - 1 = 2)$ and then dividing this number by 2 as every pair would have been counted twice. This means that we have to check $\frac{3\times 2}{2} = 3$ pairs of people. If there are 23 people, you would have to check $\frac{23\times 22}{2} = 506$ people.